space curve

$\vec{r} : \underset{\substack{\uparrow \\ \text{Interval}}}{I} \longrightarrow \mathbb{R}^n$

The limit of $\vec{r}(t)$ is computed componentwise

Ex : compute $\lim_{t\to\infty} \left\langle \frac{1+t^2}{1-t^2}, \arctan t, \frac{1-e^{-2t}}{t} \right\rangle$

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{1+t^2}{1-t^2} = \lim_{t\to\infty} \frac{\frac{1}{t^2}+1}{\frac{1}{t^2}-1} = \frac{0+1}{0-1} = -1$$

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \arctan t = \frac{\pi}{2}$$

oops, chris made a mistake. Limit not indeterminate...

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} \frac{1-e^{-2t}}{t} = 0$$

$$\therefore \lim_{t\to\infty} \left\langle \frac{1+t^2}{1-t^2}, \arctan(t), \frac{1-e^{-2t}}{t} \right\rangle = \left\langle -1, \frac{\pi}{2}, 0 \right\rangle$$

Def : A space curve is $\vec{r}(t)$ is continuous ("cts") at $t=a$ when $\lim_{t\to a} r(t) = r(a)$ < same definition as in calculus >

Ex : where is $\vec{r}(t) = \left\langle \frac{1+t^2}{1-t^2}, \arctan(t), \frac{1-e^{-2t}}{t} \right\rangle$ continuous?

NB. $\vec{r}(t)$ is cts at a iff each of $x(t)$, ~~y~~ $y(t)$, $z(t)$ is cts at a

Sol : $x(t)$ is cts when $1-t^2 \neq 0 \Rightarrow t \neq \pm 1$

so $t \in (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

$y(t)$ is cts on $(-\infty, \infty) \ni t$

$z(t)$ is cts on $(-\infty, 0) \cup (0, \infty)$

$\vec{r}(t)$ is cts on $(-\infty, -1) \cup (-1, 0) \cup (0, 1) \cup (1, \infty)$

derivative

Def. The derivative of space curve $\vec{r}(t)$ at $t=a$ is $\vec{r}'(a) = \frac{dr}{dt}\Big|_{t=a} = \lim_{h\to 0} \frac{\vec{r}(a+h)-\vec{r}(a)}{h}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Ex: compute $\vec{r}'(t)$ for $\vec{r}(t) = \langle t, t^2, \sqrt{t} \rangle$

Sol: $\vec{r}'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h)-\vec{r}(t)}{h}$

$$= \lim_{h\to 0} \frac{1}{h}\left(\langle t+h, (t+h)^2, \sqrt{t+h} \rangle - \langle t, t^2, \sqrt{t} \rangle\right)$$

$$= \lim_{h\to 0} \frac{1}{h} \langle t+h-t,\ (t+h)^2-t^2,\ \sqrt{t+h}-\sqrt{t} \rangle$$

$$= \lim_{h\to 0} \frac{1}{h} \langle h,\ h^2+2ht,\ \sqrt{t+h}-\sqrt{t} \rangle$$

$$= \langle \lim_{h\to 0} \frac{h}{h},\ \lim_{h\to 0} \frac{h^2+2ht}{h},\ \lim_{h\to 0} \frac{\sqrt{t+h}-\sqrt{t}}{h} \rangle$$

$$\lim_{h\to 0} \frac{\sqrt{t+h}-\sqrt{t}}{h} = \lim_{h\to 0} \frac{\sqrt{t+h}-\sqrt{t}}{h} \cdot \frac{\sqrt{t+h}+\sqrt{t}}{\sqrt{t+h}+\sqrt{t}}$$

$$= \lim_{h\to 0} \frac{t+h-t}{h(\sqrt{t+h}+\sqrt{t})} = \lim_{h\to 0} \frac{1}{\sqrt{t+h}+\sqrt{t}} = \frac{1}{2\sqrt{t}}$$

$$\vec{r}'(t) = \langle 1,\ 2t,\ \frac{1}{2\sqrt{t}} \rangle$$

What really happened? ($n=2$ for illustration)

$$\lim_{h\to 0} \frac{\vec{r}(t+h)-\vec{r}(h)}{h} = \lim_{h\to 0} \langle \frac{x(h+t)-x(t)}{t},\ \frac{y(h+t)-y(t)}{t} \rangle$$

$$= \langle \lim_{h\to 0} \frac{x(h+t)-x(t)}{t},\ \lim_{h\to 0} \frac{y(h+t)-y(t)}{t} \rangle$$

$$= \langle x'(t),\ y'(t) \rangle$$

Point : can also compute derivative componentwise

Properties of derivative of space curves :

Let $\vec{r}(t)$, $\vec{s}(t)$ be space curve and Let $c(t)$ be a scalar function, provided that corresponding derivative exist :

① $\frac{d}{d(t)}[\vec{r}(t)+\vec{s}(t)] = \frac{d\vec{r}}{dt} + \frac{d\vec{s}}{dt} = r'(t)+s'(t)$ sum rule in each component

② $\frac{d}{dt}[c(t)\vec{r}(t)] = C'(t)\vec{r}(t) + c(t)\vec{r}'(t)$ product rule in each component

③ $\frac{d}{dt}[r'(t)\cdot s'(t)] = \vec{r}'(t)\vec{s}(t) + \vec{r}(t)\cdot\vec{s}'(t)$ dot product rule

$\frac{d}{dt}[<x(t), g(t)> \cdot <a(t)\cdot b(t)>]$

$\frac{d}{dt}[\ x(t)a(t) + y(t)b(t)\ ]$

$\frac{d}{dt}[x(t)a(t)] + \frac{d}{dt}[y(t)b(t)]$

$= [x'(t)a(t) + a'(t)x(t)] + [y'(t)b(t) + b'(t)y(t)]$

$= [x'(t)a(t) + y'(t)b(t)] + [a'(t)x(t) + b'(t)y(t)]$

$= <x', y'> \cdot <a, b> + <a', b'> \cdot <x, y>$

④ $\frac{d}{dt}[\vec{r}(t)\times\vec{s}(t)] = \vec{r}'(t)\times\vec{s}(t) + \vec{r}(t)\times\vec{s}'(t)$ cross product

cross product not ~~com~~ commutative, order matter.

⑤ $\frac{d}{dt}[\vec{r}(c(t))] = \vec{r}'(c(t))\cdot c'(t)$ chain rule

Those are all analogous to what we learned in Calculus I

Exercise: verify each of these for space curves in R^3.

Def $\vec{r}'(t)$ is the tangent vector to $r(t)$ at time t

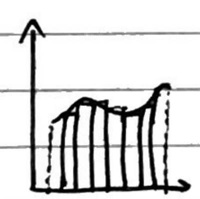
The unit tangent vector is $\frac{\vec{r}'(t)}{|\vec{r}'(t)|}$ provided $\vec{r}'(t) \neq \vec{0}$

The speed of $\vec{r}(t)$ is $|\vec{r}'(t)|$

Exercise: Prove that if $\vec{r}(t)$ has constant speed then $\vec{r}(t)$ and $\vec{r}'(t)$ are orthogonal.

Integrals of space curves:

Def $\int_{t=a}^{b} \vec{r}(t)\,dt = \langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \rangle$



$\int_a^b f(t)\,dt = \lim_{n\to\infty} \sum_{j=0}^{n} f(t_a^*)\Delta t_a$ calculus I

Same formula works for space curve.

for space curve $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

Interpretation: Just like Calculus I

$\int_a^b \vec{r}(t)\,dt$ represents "displacement"

Arc length

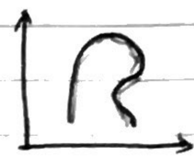
The arc length of a curve should be computable by

① approximate curve by straight line segment

② length of each segment

adds the approx. length of curve

Using more and more line segments



Successive approximation limit to tangent line.

Point : arc length on [a, b] of $\vec{r}(t)$ is

$$S = \int_a^b |\vec{r}\,'(t)|\,dt$$